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Generating function for Clebsch–Gordan coefficients, of the $\mathrm{su}_q(2)$ quantum algebra

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Abstract. Some methods have been developed to calculate the $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients (CGC). Here we develop a method based on the calculation of Clebsch–Gordan generating functions through the use of 'quantum algebraic' coherent states. Calculating the $\mathrm{su}_q(2)$ CGC by means of this generating function is an easy and straightforward task.

1. Introduction

Quantum algebras have been extensively used in the literature for different purposes, in different areas of interest [1]. They are also known as quantum universal enveloping algebras and are mathematically not less than Hopf algebras.

The study of coherent states associated with quantum algebras for the q-harmonic oscillator has been established some time ago [2] and a resolution of unity for the q-analogue oscillator coherent states has already been found through the use of the definitions of q-differentiation and q-integration, as seen in [3,4]. For the $\mathrm{su}_q(2)$ algebra, q-analogues of coherent states [5] have been obtained and they are shown to have a resolution of unity related to the q-differential calculus [6].

The aim of this work is to calculate a generating function for the $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients. The method we use here is based on the idea developed in [7] for the usual $\mathrm{su}(2)$ Clebsch–Gordan coefficients. For calculating the $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients, other methods have also been developed, namely, some algebraic methods [8,9] and a method based on the basic hyper-geometric functions [10]. The underlying idea for the following derivation of the Clebsch–Gordan generating function is extremely simple and yields very easily handled expressions. Once the generating functions are obtained it becomes trivial to write the *wanted* $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients down, as will be shown later in the present paper.

2. $\mathrm{su}_q(2)$ coherent states

We begin this section by giving some definitions and expressions which will be vital to the development of our method. For the sake of completeness and simplicity in the main text, some formulae are given in appendix 1.

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The generators of the $su_q(2)$ algebra obey the following commutation relations [2]

$$\begin{aligned}[][J_0, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= [2J_0]\end{aligned} \tag{1}$$

where the q-number $[x]$ is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

The above operators, when applied to a basis $|j\ m\rangle$ of the carrier space V^j of the representation T^j of $su_q(2)$ yield

$$\begin{aligned} J_0|j\ m\rangle &= m|j\ m\rangle \\ J_\pm|j\ m\rangle &= ([j \mp m][j \pm m + 1])^{1/2}|j\ m \pm 1\rangle. \end{aligned} \tag{3}$$

with $m = -j, -j+1, \ldots, j$ and $j = 0, 1/2, 1, \ldots$.

The q-analogues of the su(2) coherent states are usually written as [6]

$$|z\rangle = e_q^{\bar{z}J_+}|j\ -j\rangle = \sum_{m=-j}^{j} C_{jm}\bar{z}^{j+m}|j\ m\rangle \tag{4}$$

where

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}. \tag{5}$$

and

$$C_{jm} = \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2} \tag{6}$$

the q-binomial coefficient being defined in appendix 1.

For this representation, it can be shown that the resolution of unity is [6]

$$I = \int \mathrm{d}\mu(z)|z\rangle\langle z| \tag{7}$$

the measure being

$$\mathrm{d}\mu(z) = \frac{[2j+1]}{\pi}\frac{1}{[1(+)|z|^2]^{2j+2}}\mathrm{d}^2 z \tag{8}$$

where $[x(\pm)y]^m$ is defined in appendix 1 and $\mathrm{d}^2 z = \frac{1}{2}\mathrm{d}\theta\, \mathrm{d}_q|z|^2$, the integration over θ running from 0 to 2π and the q-integration over $|z|^2$ from 0 to infinity.

Given an arbitrary state $|\phi\rangle$ in V^j and due to the existence of the resolution of unity, one may find a holomorphic (or q-analogue Bargmann [11]) representation that reads

$$\phi(z) = \langle z|\phi\rangle \tag{9}$$

with the standard $su_q(2)$ basis given by

$$\phi_{jm}(z) = \langle z|j\ m\rangle = C_{jm}\ z^{j+m}. \tag{10}$$

The $su_q(2)$ operators in the holomorphic representation, which obey the commutation relations given by (1) are [6]

$$\begin{aligned} J_0 &= z\partial/\partial z - j \\ J_+ &= z[2j - z\partial/\partial z] \\ J_- &= D_z \end{aligned} \tag{11}$$

where D_z is the q-derivative [3] such that

$$D_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}. \tag{12}$$

Observing (4), it follows that the scalar product between two coherent states $|z\rangle$ and $|\chi\rangle$ is

$$\langle\chi|z\rangle = [1\ (+)\ \chi\bar{z}]^{2j} \tag{13}$$

Given an arbitrary state $|\phi\rangle >$ in V^j, such that $\phi(\chi) = \langle\chi|\phi\rangle$ and utilizing (7) and (13), the reproducing kernel is shown to be

$$k(\chi, z) = [1\ (+)\ \chi\bar{z}]^{2j} \tag{14}$$

and therefore

$$\phi(\chi) = \int \mathrm{d}\mu(z)\, k(\chi, z)\phi(z). \tag{15}$$

From now on, all calculations are performed in the q-deformed space.

3. Vector addition of angular momenta

The total angular momentum of an $su_q(2)$ system consisting of two sub-systems $j = j_1 + j_2$, where j_1 and j_2 are the angular momenta of sub-systems 1 and 2 respectively, such that $|j_1 - j_2| \leqslant j \leqslant j_1 + j_2$ is given by [8, 9]

$$\begin{aligned} J_0(12) &= J_0(1) \otimes I(2) + I(1) \otimes J_0(2) \\ J_\pm(12) &= J_\pm(1) \otimes q^{J_0(2)} + q^{-J_0(1)} \otimes J_\pm(2) \end{aligned} \tag{16}$$

where $J_0(12)$ and $J_\pm(12)$ obey the following commutation relations:

$$\begin{aligned} [J_0(12), J_\pm(12)] &= \pm J_\pm(12) \\ [J_+(12), J_-(12)] &= [2J_0(12)]. \end{aligned} \tag{17}$$

Notice that $J_0(i)$ and $J_\pm(i)$ are the operators defined in (11), where for $i = 1$, z and j become z_1 and j_1 and for $i = 2$, they become z_2 and j_2. The same modifications hold in (9) and (10).

4. Coherent states in the space $V^{j_1} \otimes V^{j_2}$

The uncoupled basis in the space $V^{j_1} \otimes V^{j_2}$ has the form $|j_1 m_1\rangle |j_2 m_2\rangle$ and its representation is $\phi_{j_1 m_1}(z_1)\phi_{j_2 m_2}(z_2)$. It is well known that the direct product of representations $T^{j_1} \otimes T^{j_2}$ can be decomposed as a direct sum $T^{j_1} \otimes T^{j_2} = \sum_j \oplus T^j$ [12] where $|j_1 - j_2| \leqslant j \leqslant j_1 + j_2$. In the carrier space V^j of T^j, the coherent state is defined to be

$$|z\rangle = e_q^{\bar{z} J_+(12)} |j \ -j\rangle \tag{18}$$

in analogy with (4), where $|j \ -j\rangle$ stands for the lowest weight state in the V^j space. From the above considerations, it is straightforwardly shown that the state obeying the conditions $J_-(12)\phi_{j,-j}(z_1, z_2) = 0$ and $J_0(12)\phi_{j,-j}(z_1, z_2) = -j\phi_{j,-j}(z_1, z_2)$ is

$$\phi_{j,-j}(z_1, z_2) = C_j \ [z_1 \ (-) \ q^{-(j+1)} z_2]^{j_1+j_2-j} \tag{19}$$

where

$$C_j = \left(\frac{[2j_1]![2j_2]![2j+1]!}{[j_1+j_2-j]![j_1-j_2+j]![j-j_1+j_2]![j_1+j_2+j+1]!} \right)^{1/2} \times q^{(j_1+j_2-j)(j+1+j_1-j_2)/2} \tag{20}$$

is the normalization constant. In the notation used for this calculation, $\phi_{j,-j}(z_1, z_2)$ is defined as

$$\phi_{j,-j}(z_1, z_2) = \langle z_1 \ z_2 | j - j\rangle$$

where $|z_1 \ z_2\rangle = |z_1\rangle|z_2\rangle$ with $|z_1\rangle$ being the coherent state in the space V^{j_1} and $|z_2\rangle$ the coherent state in V^{j_2}.

In the above calculations, the q-derivatives written in appendix 1 have been used.

5. Generating functions

From the definitions for the coherent states in the V^j space, vide (18) and in the $V^{j_1} \otimes V^{j_2}$ space, we obtain

$$\langle z_1 \ z_2 | \chi\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m=-j}^{j} \begin{bmatrix} 2j_1 \\ j_1+m_1 \end{bmatrix}^{1/2} \begin{bmatrix} 2j_2 \\ j_2+m_2 \end{bmatrix}^{1/2} \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2} \times z_1^{j_1+m_1} \ z_2^{j_2+m_2} \ \bar{\chi}^{j+m} \langle j_1 \ m_1 \ j_2 \ m_2 \ | j \ m\rangle_q \tag{21}$$

and we are left with the calculation of $\langle z_1 \ z_2|\chi\rangle$, i.e. the generating function from which the Clebsch–Gordan coefficients appearing in the right-hand side of the above expression can be easily obtained by q-differentiation, as is discussed later. One has to bear in mind that $|z_1\rangle|z_2\rangle|z_1 \ z_2\rangle$ is the coherent state in the $V^{j_1} \otimes V^{j_2}$ space and $|\chi\rangle$ is the coherent state coming from the V^j space. Using the definition for the resolution of unity in space $V^{j_1} \otimes V^{j_2}$ (which follows in a straightforward way from (7) and (8), we can write

$$\langle z_1\ z_2|\chi\rangle = \int \mathrm{d}\mu(\xi_1)\,\mathrm{d}\mu(\xi_2)\,\langle z_1|\xi_1\rangle\langle z_2|\xi_2\rangle\langle \xi_1\ \xi_2|e_q^{\bar{\chi}J_+^{(12)}}|j\ \ -j\rangle \quad (22)$$

where $\mathrm{d}\mu(\xi_1)$ and $\mathrm{d}\mu(\xi_2)$ are the measures in spaces V^{j_1} and V^{j_2}, respectively (see (8)). With the help of (14), (15) we obtain

$$\langle z_1\ z_2|\chi\rangle = e_q^{\bar{\chi}J_+^{(12)}}\phi_{j,-j}(z_1, z_2). \quad (23)$$

From the fact that [13]

$$e_q^{\bar{\chi}J_+^{(12)}}|j_1\ m_1\rangle|j_2\ m_2\rangle = e_q^{\bar{\chi}q^{m_2}J_+^{(1)}}|j_1\ m_1\rangle\ e_q^{\bar{\chi}q^{-m_1}J_+^{(2)}}|j_2\ m_2\rangle \quad (24)$$

and

$$e_q^{\lambda J_+}\ z^n = [1\ (+)\ \lambda z]^{2j-n} \quad (25)$$

$\langle z_1\ z_2|\chi\rangle$ becomes

$$\begin{aligned}\langle z_1\ z_2|\chi\rangle = C_j \sum_{m=0}^{j_1+j_2-j} &\begin{bmatrix} j_1+j_2-j \\ m \end{bmatrix} (-q^{-(j+1)})^m z_1^{(j_1+j_2-j-m)} z_2^m \\ &\times [1\ (+)\ \bar{\chi}q^{(m-j_2)}z_1]^{(j_1-j_2+j+m)}[1\ (+)\ \bar{\chi}q^{(j-j_2+m)}z_2]^{(2j_2-m)}. \quad (26)\end{aligned}$$

With the use of the identity [13]

$$[1\ (+)\ q^a z]^r[1\ (+)\ q^{r+s+a}z]^s = [1\ (+)\ q^{s+a}z]^{r+s} \quad (27)$$

(26) reads

$$\begin{aligned}\langle z_1\ z_2|\chi\rangle = C_j\ &[1\ (+)\ \bar{\chi}q^{-j_2}z_1]^{(j_1-j_2+j)}\ [1\ (+)\ \bar{\chi}q^{j_1}z_2]^{(j_2-j_1+j)} \\ &\times \sum_{m=0}^{j_1+j_2-j} \begin{bmatrix} j_1+j_2-j \\ m \end{bmatrix} (-q^{-(j+1)})^m z_1^{(j_1+j_2-j-m)} z_2^m \\ &\times [1\ (+)\ \bar{\chi}q^{(j_1-2j_2+j+m)}z_1]^m \\ &\times [1\ (+)\ \bar{\chi}q^{(j_1-2j_2+m)}z_2]^{(j_1+j_2-j-m)}. \quad (28)\end{aligned}$$

Finally, another simplification can be performed with the help of the expression

$$\sum_{m=0}^{J} \begin{bmatrix} J \\ m \end{bmatrix} (-1)^m [x\ (+)\ yq^{m-1}]^m [z\ (+)\ yq^m]^{J-m} = [z\ (-)\ x]^J \quad (29)$$

yielding

$$\begin{aligned}\langle z_1\ z_2|\chi\rangle = C_j\ &[1\ (+)\ \bar{\chi}q^{-j_2}z_1]^{(j_1-j_2+j)}[1\ (+)\ \bar{\chi}q^{j_1}z_2]^{(j_2-j_1+j)} \\ &\times [z_1\ (-)\ q^{-(j+1)}z_2]^{(j_1+j_2-j)} \quad (30)\end{aligned}$$

which is the Clebsch–Gordan generating function we are looking for. Obtaining a general formula for the Clebsch–Gordan coefficient is now a simple task. From (21) we can write

$$
\begin{aligned}
\langle j_1\ m_1\ j_2\ m_2|j\ m\rangle_q
&= \left(\frac{[j_1-m_1]!}{[2j_1]![j_1+m_1]!}\right)^{1/2}\left(\frac{[j_2-m_2]!}{[2j_2]![j_2+m_2]!}\right)^{1/2}\left(\frac{[j-m]!}{[2j]![j+m]!}\right)^{1/2} \\
&\quad \times D_{z_1}^{j_1+m_1} D_{z_2}^{j_2+m_2} D_{\bar{\chi}}^{j+m} \langle z_1\ z_2|\chi\rangle|_{z_1=z_2=\bar{\chi}=0}.
\end{aligned}
\tag{31}
$$

With the help of a q-analogue of Leibnitz's rule given in appendix 1 (39), we finally obtain

$$
\begin{aligned}
\langle j_1\ m_1\ j_2\ m_2|j\ m\rangle_q
&= C_j \sqrt{\frac{[j_1-m_1]![j_1+m_1]![j_2-m_2]![j_2+m_2]![j-m]![j+m]!}{[2j_1]![2j_2]![2j]!}} \\
&\quad \times q^{j_1(j_2+m_2)-j_2(j+m_1-j_2)}\ [j_1+j_2-j]![j_1-j_2+j]![j_2-j_1+j]! \\
&\quad \times \sum_k (-1)^k\ q^{-(j_1+j_2+j+1)k}[k]!^{-1}[j_1+j_2-j-k]!^{-1}[j-j_1-m_2+k]!^{-1} \\
&\quad \times [j-j_2+m_1+k]!^{-1}[j_1-m_1-k]!^{-1}[j_2+m_2-k]!-1
\end{aligned}
$$

which is a general expression for $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients. This expression agrees with (47) of Groza *et al* [10].

We remark, however, that calculating Clebsch–Gordan coefficients is often easier if calculation is performed by substituting the 'wanted' values in (31) and then q-differentiating the obtained expressions instead of using the above general formula, specially if an 'algebraic manipulator program' is available. Simple examples of how to obtain the Clebsch–Gordan coefficient itself by means of q-differentiation are worked out in appendix 2.

6. Conclusion

In this paper we have developed a way of calculating a $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients generating function through the use of coherent states in a holomorphic (or q-analogue Bargmann) representation. The advantage of this method is that the generating function is obtained in a straightforward way. With the help of q-differentiation, the $\mathrm{su}_q(2)$ Clebsch–Gordan coefficients are easily worked out from the generating function, as shown in appendix 2.

It is worth pointing out that for $q = 1$, the generating function written in (30) becomes precisely expression (14), given by Belissard and Holtz [7] for the su(2) algebra.

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Appendix 1

Some formulae mentioned in the main text are shown below. The expression

$$[a\ (\pm)\ b]^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} a^{m-k}(\pm b)^k \tag{32}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[m-n]![n]!} \tag{33}$$

is the q-binomial definition. Some q-derivatives are

$$D_z(e_q^{az}) = a\ e_q^{az} \tag{34}$$

$$D_z\ z^n = [n]\ z^{n-1} \tag{35}$$

$$D_z(f(z)g(z)) = (D_z f(z))g(qz) + f(q^{-1}z)D_z g(z) \tag{36}$$

$$D_{z_2}[az_1(\pm)bz_2]^m = \pm[m]b[az_1(\pm)bz_2]^{m-1} \tag{37}$$

where a and b are constants and $[az_1(\pm)bz_2]^m$ is defined in (32) above. From (36), and the following property:

$$D_z f(qz) = q\ D_z f(z)|_{z=qz} \tag{38}$$

one can show that the q-analogue Leibnitz's rule is given by

$$D_z^n f(z)g(z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} D_z^{n-k} f(z)|_{z=q^{-k}z}\ D_z^k g(z)|_{z=q^{n-k}z}. \tag{39}$$

Appendix 2

Here we show two simple examples of how to obtain the Clebsch–Gordan coefficients by q-differentiating the generating function.

Example 1. Let's calculate $\langle j_1\ m\ 2\ 0|j_1+2\ m\rangle_q$, a necessary coefficient when calculating quadrupole transition probabilities in a large number of atomic and nuclear models [14]. Substituting the correct values for j_1, m_1, j_2, m_2, j and m in (30) we have

$$\langle z_1\ z_2|\chi\rangle = C_j\ [1(+)\bar{\chi}q^{-2}z_1]^{2j_1}\ [1(+)\bar{\chi}q^{j_1}z_2]^4. \tag{40}$$

By q-differentiating the above expression, we obtain

$$D_{\bar{\chi}}^{j_1+2+m} D_{z_2}^2 D_{z_1}^{j_1+m}\langle z_1\ z_2|\chi\rangle|_{z_1=z_2=\bar{\chi}=0} = C_j \frac{[4][3][2j_1]![j_1+m+2]!}{[j_1-m]!} q^{-2m} \tag{41}$$

where $C_j = 1$. Finally, substituting the above expression in (31), we obtain

$$\langle j_1 m2\ 0|j_1+2\ m\rangle_q$$

$$= q^{-2m}\sqrt{\frac{[4][3][j_1-m+2][j_1-m+1][j_1+m+2][j_1+m+1]}{[2][2j_1+4][2j_1+3][2j_1+2][2j_1+1]}}$$

which is precisely the expression found in table 4 of [8].

Example 2. The $\langle j_1\ m_1\ 1/2\ 1/2|j_1 - 1/2\ m\rangle_q$ coefficient, where $m = m_1 + 1/2$ is calculated below. Again, from (30) we may write

$$\langle z_1\ z_2|\chi\rangle = C_j\ [1(+)\bar{\chi}q^{-1/2}z_1]^{2j_1-1}[z_1(-)q^{-(j_1+1/2)}z_2] \tag{42}$$

from where we calculate

$$D_{\bar{\chi}}^{j_1+m_1}D_{z_2}^{1}D_{z_1}^{j_1+m_1}\langle z_1\ z_2|\chi\rangle|_{z_1=z_2=\bar{\chi}=0} = -C_j\frac{[2j_1]![j_1+m_1]!}{[j_1-m_1-1]!}q^{-1/2(j_1+m_1)-(j_1+1/2)}$$

where

$$C_j = q^{j_1}\frac{[2j_1]!}{\sqrt{[2j_1-1]![2j_1+1]!}} \tag{43}$$

and hence

$$\langle j_1\ m_1\ 1/2\ 1/2|j_1 - 1/2\ m\rangle_q = -q^{-1/2(j_1+m+1/2)}\sqrt{\frac{[j_1-m+1/2]}{[2j_1+1]}} \tag{44}$$

which is the $su_q(2)$ Clebsch–Gordan coefficient also found in table 1 of [8].

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